First Derivative Test for Local Extrema

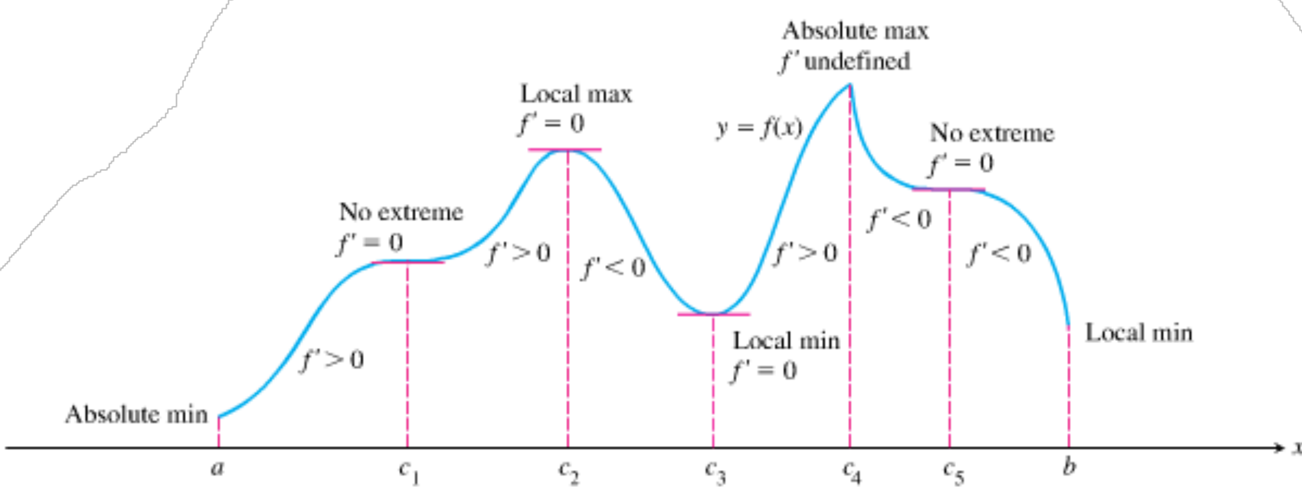


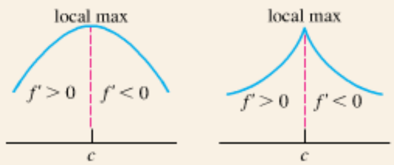
Figure 4.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c:

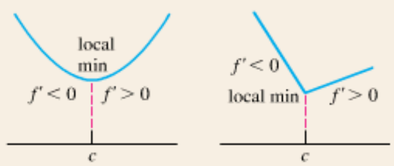
1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

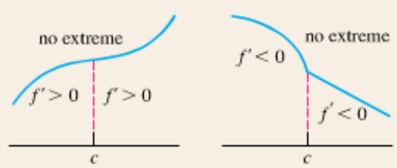
continued

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

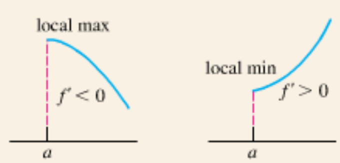
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

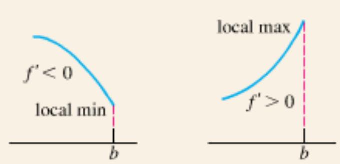
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$ **(b)** $g(x) = (x^2 - 3)e^x$

$-8 + 24 - 5$

① find critical points where $f'(x) = 0$ and/or $f'(x)$ is undefined

② Test points on either side of critical points in 1st derivative

Test point on right side of a left endpt.

Test point on left side of a right endpt.

a) $f'(x) = 3x^2 - 12$

critical pts $x = 2$, $x = -2$

$f'(x) = 0$ $\quad 3x^2 - 12 = 0$

$3(x^2 - 4) = 0$

$3(x-2)(x+2) = 0$

② $+ \mid - \mid +$ on a number line with -2 and 2 marked

at $x = -2$ local max $= 11$

at $x = 2$ local min $= -21$

$\boxed{f'(x) < 0 \ (-2, 2) \quad f'(x) > 0 \ (-\infty, -2) \ (2, \infty)}$

b) $(x^2 - 3)e^x$

$f'(x) = (x^2 - 3)e^x + e^x(2x)$

$= e^x(x^2 + 2x - 3) = 0$

$f'(x) = 0$ at $(x+3)(x-1) = 0$

$x = -3$ $\quad x = 1$

$+ \mid - \mid +$ on a number line with -3 and 1 marked

$f'(x) > 0 \ (-\infty, -3) \ (1, \infty)$

at $x = -3$ local max $= .299$

at $x = 1$ local min $= -5.437$

$f'(x) < 0 \ (-3, 1)$

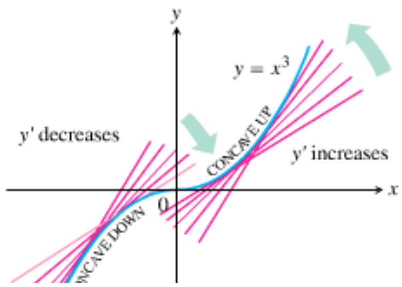
Concavity

DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if y' is increasing on I.

(b) **concave down** on an open interval I if y' is decreasing on I.



Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$.

(b) concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

(a) $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

a) $y' = 2x$
$y'' = 2$

$y'' > 0$ always
therefore, always concave up

b) $y' = \cos x$
$y'' = -\sin x$

$y'' < 0$ $(0, \pi)$ concave down
$y'' > 0$ $(\pi, 2\pi)$ concave up

Points of Inflection

DEFINITION **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

$$y' = e^{-x^2} \cdot -2x \quad = -2xe^{-x^2}$$

$$y'' = -2x\left(e^{-x^2} \cdot -2x\right) + e^{-x^2} \cdot -2$$

$$= 4x^2e^{-x^2} - 2e^{-x^2}$$

$$= 2e^{-x^2}(2x^2-1)$$

$$2x^2-1=0$$

$$2x^2=1$$

$$x^2=\frac{1}{2}$$

$$x=-\frac{1}{\sqrt{2}} \qquad x=\frac{1}{\sqrt{2}}$$

$$e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$e^{-\frac{1}{2}}$$

$+ \quad | \quad - \quad | \quad +$

$-\frac{1}{\sqrt{2}} \qquad \frac{1}{\sqrt{2}}$

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right) \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$$

EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function f on the interval $[-4, 4]$ is shown in Figure 4.25. Answer the following questions about f, justifying each answer with information obtained from the graph of f'.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave up?

(c) At which x-coordinates does f have local extrema?

(d) What are the x-coordinates of all inflection points of the graph of f?

(e) Sketch a possible graph of f on the interval $[-4, 4]$.

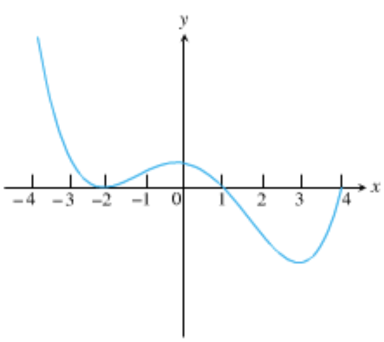
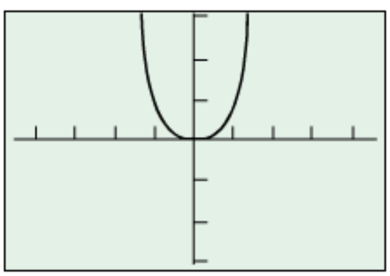


Figure 4.25 The graph of f', the derivative of f, on the interval $[-4, 4]$.

Caution: It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

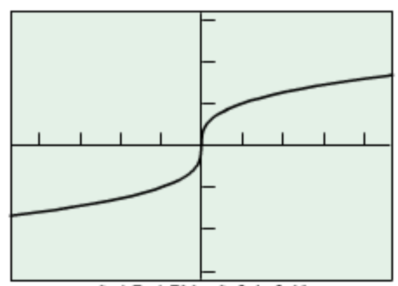
1. *The second derivative can be zero at a noninflection point*. For example, consider the function $f(x) = x^4$ (Figure 4.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, (0, 0) is not an inflection point. Note that f'' does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point*. For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 4.28). The concavity changes at $x = 0$, but there is a *vertical* tangent line, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.



[– 4.7, 4.7] by [–3.1, 3.1]

Figure 4.27 The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



[–4.7, 4.7] by [–3.1, 3.1]

Figure 4.28 The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

EXAMPLE 5 Studying Motion along a Line

A particle is moving along the x-axis with position function

$$x(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Table 4.2 Population of Alaska

Years since 1900	Population
20	55,036
30	59,278
40	75,524
50	128,643
60	226,167
70	302,583
80	401,851
90	550,043
100	626,932

Source: Bureau of the Census, U.S. Chamber of Commerce.

EXAMPLE 6 Population Growth in Alaska

Table 4.2 shows the population of Alaska in each 10-year census between 1920 and 2000.

(a) Find the logistic regression for the data.

(b) Use the regression equation to predict the Alaskan population in the 2020 census.

(c) Find the inflection point of the regression equation. What significance does the inflection point have in terms of population growth in Alaska?

(d) What does the regression equation indicate about the population of Alaska in the long run?

Second Derivative Test for Local Extrema

THEOREM 5 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

EXAMPLE 8 Using f' and f'' to Graph f

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

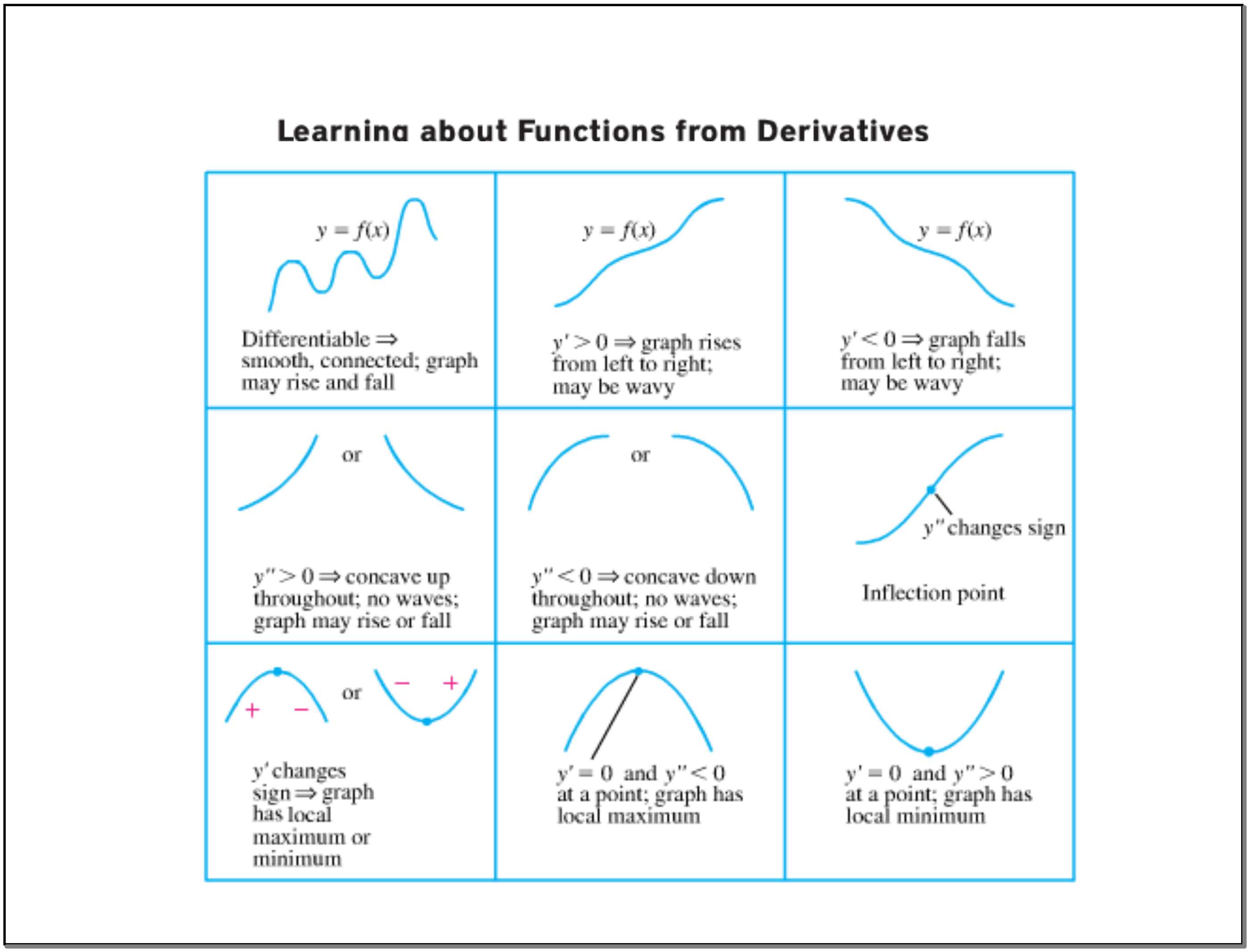
(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch a possible graph for f.

EXPLORATION 1 **Finding f from f'**

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.


Learning about Functions from Derivatives
y = f(x)
Differentiable ⇒ smooth, connected; graph may rise and fall
y = f(x)
y′ > 0 ⇒ graph rises from left to right; may be wavy
y = f(x)
y′ < 0 ⇒ graph falls from left to right; may be wavy
or
y″ > 0 ⇒ concave up throughout; no waves; graph may rise or fall
or
y″ < 0 ⇒ concave down throughout; no waves; graph may rise or fall
y″ changes sign
Inflection point
+ − or − +
y′ changes sign ⇒ graph has local maximum or minimum
y′ = 0 and y″ < 0 at a point; graph has local maximum
y′ = 0 and y″ > 0 at a point; graph has local minimum

EXAMPLE 9 Analyzing a Discontinuous Derivative

A function f is continuous on the interval $[-4, 4]$. The discontinuous function f', with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the x-coordinates of all local extrema and points of inflection of f.

(b) Sketch a possible graph of f.

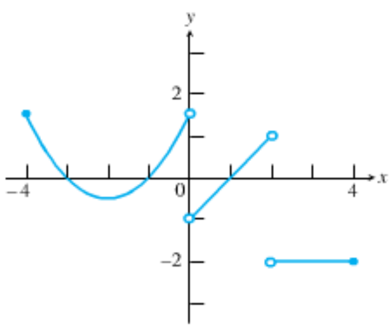


Figure 4.33 The graph of f', a discontinuous derivative.

EXPLORATION 2 **Finding f from f' and f''**

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	$+$	does not exist	$-$	0	$-$
f''	$+$	does not exist	$+$	0	$-$

1. Find where all absolute extrema of f occur.
2. Find where the points of inflection of f occur.
3. Sketch a possible graph of f.